

# Families of traveling impulses and fronts in some models with cross-diffusion

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## Abstract

An analysis of traveling wave solutions of partial differential equation (PDE) systems with cross-diffusion is presented. The systems under study fall in a general class of the classical Keller-Segel models to describe chemotaxis. The analysis is conducted using the theory of the phase plane analysis of the corresponding wave systems without a priori restrictions on the boundary conditions of the initial PDE. Special attention is paid to families of traveling wave solutions. Conditions for existence of front-impulse, impulse-front, and front-front travelling wave solutions are formulated. In particular, the simplest mathematical model is presented that has an impulse-impulse solution; we also show that a non-isolated singular point in the ordinary differential equation (ODE) wave system implies existence of free-boundary fronts. The results can be used for construction and analysis of different mathematical models describing systems with chemotaxis.

**Keywords:** Keller-Segel model, traveling wave solutions, cross-diffusion

## 1 Introduction

In this paper we study one-dimensional traveling wave solutions for the models in the form

$$\begin{aligned} U_t &= (\mu U_x - f(U, V)V_x)_x, \\ V_t &= g(U, V). \end{aligned} \tag{1}$$

Here  $\mu > 0$  is a constant;  $f(U, V)$  and  $g(U, V)$  are functions whose properties will be specified later;  $U = U(x, t)$ ,  $V = V(x, t)$ ; in the following we put  $\mu = 1$  without loss of generality.

The model (1) is known as a particular case of the classical Keller-Segel models to describe chemotaxis, the movement of a population  $U$  to a chemical signal  $V$  (see, e.g., [1, 2, 3]). In system (1)  $\mu$  denotes the constant diffusion coefficient;  $f(U, V)/U$  is the chemotactic sensitivity, which can be either positive or negative;  $g(U, V)$  describes production and degradation of the chemical signal; it is customary to include also in the second equation of (1) the diffusion term of the form  $DV_{xx}$  which would describe diffusion of the chemical signal, but, we adopt hereafter, that,

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to a first approximation,  $D$  can be taken zero. For biological interpretation of the solutions of (1) we refer to the cited literature, references therein, and to Section 3.2; macroscopic derivation of equation (1) can be found in, e.g., [4].

The chemotactic models are the partial differential equations (PDEs) with cross-diffusion terms; these systems possess special mathematical peculiarities [5]. Such systems were used, e.g., to model the movement of traveling bands of *E. coli* [2, 3, 6], amoeba clustering [7], insect invasion in a forest [8], species migration [9], tumor encapsulation and tissue invasion [10, 11]. Many different spatially non-homogeneous patterns can be observed in chemotactic models, for a survey see, e.g., [12] and references therein. One such pattern is that of traveling waves which spread through the population.

A *traveling wave* is a bounded solution of system (1) having the form

$$U(x, t) = U(x + ct) \equiv u(z), \quad V(x, t) = V(x + ct) \equiv v(z),$$

where  $z = x + ct$  and  $c$  is the speed of wave propagation along  $x$ -axis;  $u(z)$  and  $v(z)$  are the wave profiles ( $u$ -profile and  $v$ -profile respectively) of solution  $(U(x, t), V(x, t))$ .

Substituting these traveling wave forms into (1) we obtain

$$\begin{aligned} cu' &= (u' - f(u, v)v')', \\ cv' &= g(u, v), \end{aligned}$$

where primes denote differentiation with respect to  $z$ .

On integrating the first equation in the last system we have *the wave system* of (1):

$$\begin{aligned} u' &= cu + f(u, s)g(u, s)/c + \alpha, \\ v' &= g(u, s)/c. \end{aligned} \tag{2}$$

Here  $\alpha$  is the constant of integration that depends on the boundary conditions for  $U(x, t)$  and  $V(x, t)$ . In various applications it is usually possible to determine this constant prior to analysis of the wave system. For instance, considering system (1) as a model of chemotactic movement [2], where the variable  $U(x, t)$  plays the role of the population density and  $V(x, t)$  is an attractant, one usually supposes that  $\int_{-\infty}^{\infty} u(x, t) dx$  should be finite, which implies that  $\alpha = 0$  (e.g., [1]). On the contrary, in our analysis we do not specify the boundary conditions for (1) and consider  $\alpha$  as a new parameter.

Each traveling wave solution of (1) has its counterpart as a bounded orbit of (2) for some  $\alpha$ ; in our study we elucidate the following question: for which  $\alpha$  there exist traveling wave solutions of (1) and describe all such solutions. We also note that the case of  $\alpha = 0$  does not exclude a model with infinite mass of  $U(x, t)$  if the traveling wave solution is a front; moreover, the solutions corresponding to finite mass can be only impulses (see below for the terminology).

It is worth noting that due to specific form of system (1) with cross-diffusion terms the wave system has the same dimension as the initial system (1), which significantly simplifies the analysis. This is one of peculiarities which distinguishes cross-diffusion PDEs from those with only diffusion terms (see also [8, 13]).

We shall study possible wave profiles of (1) and their bifurcations with changes of the parameters  $c$  and  $\alpha$  by the methods of phase plane analysis and bifurcation theory [14, 15]. In

this way, the problem of describing all traveling wave solutions of system (1) is reduced to the analysis of phase curves and bifurcations of solutions of the wave system (2) without a priori restrictions on boundary conditions for (1).

There exists a known correspondence between the bounded traveling wave solutions of the spatial model (1) and the orbits  $u(z)$ ,  $v(z)$  of the wave system (2) (e.g., [8, 16, 17, 18]) that we only list for the cases most important for our exposition.

**Proposition 1.**

- i. A wave front in  $U$  (or  $V$ ) component corresponds to a heteroclinic orbit that connects singular points of (2) with different  $u$  (or  $v$ ) coordinates (Fig.1a);
- ii. A wave impulse in  $U$  (or  $V$ ) component corresponds to a heteroclinic orbit that connects singular points with identical  $u$  (or  $v$ ) coordinates (Fig.1b) or to a homoclinic curve of a singular point  $(u, v)$  of (2) (Fig.1c).

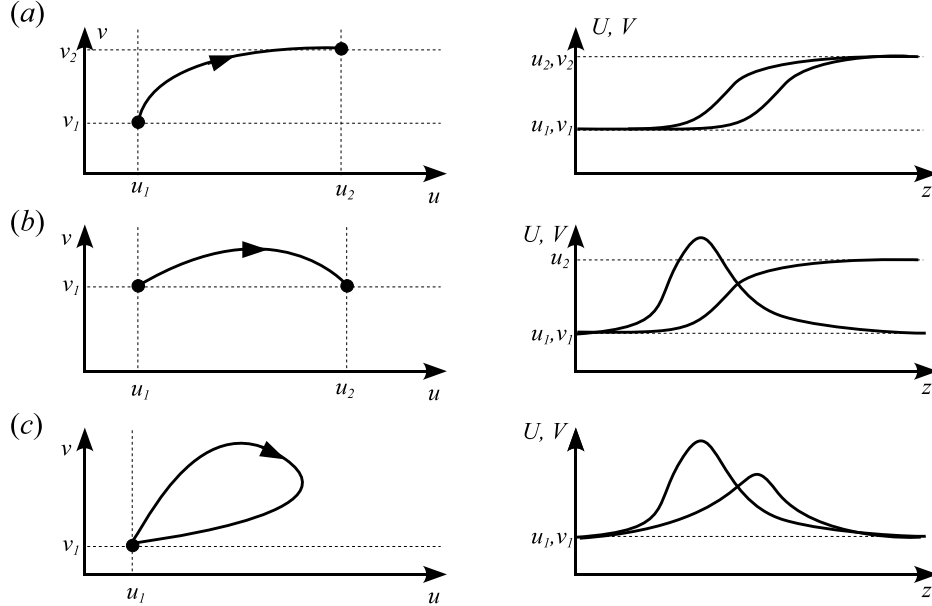


Figure 1: Correspondence between bounded traveling wave solutions of system (1) (on the right) and the phase curves of the wave system (2) (on the left); the black dots are singular points of (2). (a) A front-front solution; (b) a front-impulse solution; (c) an impulse-impulse solution

Hereinafter we shall adopt the following terminology: we define the type of a traveling wave solution of (1) with a two word definition; e.g., a front-impulse solution means that  $u$ -profile is a front, and  $v$ -profile is an impulse (the order of the terms is important).

For system (1) several results on the existence of one-dimensional traveling waves are known; see, e.g., [1, 3, 8, 9, 13, 16, 19, 20, 21, 22]. In most of these references the analysis is conducted using a particular model which is given in an explicit form. Quite a different approach was used in [1] where the authors consider more general model than (1) and do not restrict themselves to

analyzing a model with specific functions  $f(U, V)$ ,  $g(U, V)$ ; instead their aim was to understand how these functions have to be related to each other in order to result in traveling wave patterns for  $U$  and  $V$ . We consider a general class of models as well, and our task is to infer possible kinds of wave solutions under given restrictions on  $f(U, V)$  and  $g(U, V)$ .

Our main goal is as follows: we impose some constraints on the functions  $f(U, V)$ ,  $g(U, V)$  and study possible traveling wave solutions with increasing complexity of  $f$ ,  $g$ . Special attention is paid to the families of traveling wave solutions such that the corresponding wave system possesses an infinite number of bounded orbits. We present the simplest possible models in the form (1) that display traveling wave solutions of a specific kind.

The main class of the models we deal with is defined in the following way.

**Definition 1.** *We shall call model (1) the separable model if*

$$(C1) \quad f(u, v) = f_1(u)f_2(v), \quad g(u, v) = g_1(u)g_2(v),$$

where  $f_1(u)$ ,  $g_1(u)$  are smooth functions for  $a \leq u < \infty$ ;  $g_2(v)$  is smooth;  $f_2(v)$  is a rational function:

$$f_2(v) = \frac{Z(v)}{R(v)},$$

for  $b \leq v < \infty$ ; here  $a, b > -\infty$  are real constants.

The separable model will be called the reduced separable model if (C1) holds and

$$(C2) \quad f_2(v)g_2(v) \equiv \text{const.}$$

We organize the paper as follows. In Section 2 we present full classification of traveling wave solutions of the reduced separable models; we also specify necessary and sufficient conditions for these models to possess specific kinds of traveling waves. Section 3 is devoted to the analysis of the separable model; we show which types of traveling waves can be expected in addition to the types described in Section 2; we also analyze a generalized Keller-Segel model, which does not belong to the class of the separable models but display a number of similar properties together with essentially new ones. Section 4 contains discussion and conclusions; finally, the details of numerical computations are presented in Appendix.

## 2 Wave solutions of the reduced separable model

In this section we present the full classification of possible traveling wave solutions of system (1) that satisfies (C1), (C2). The reason we start with the reduced separable models is twofold. First, there are models in the literature that have this particular form (see, e.g., [20, 22, 23]); second, the special form of the wave system allows the exhaustive investigation of traveling wave solutions of (1).

The wave system of the reduced separable model reads

$$\begin{aligned} u' &= cu + f_1(u)g_1(u)/c + \alpha \equiv h(u), \\ v' &= g_1(u)g_2(v)/c, \end{aligned} \tag{3}$$

where the first equation is independent of  $v$ .

## 2.1 The cell structure of the phase plane of system (3)

We start with the case of general position. We assume that the following conditions of non-degeneracy are fulfilled (later we will relax some of these assumptions):

- (A1)  $h(u)$  has no multiple roots;
- (A2)  $g_2(v)$  has no multiple roots;
- (A3)  $g_1(u)$  and  $h(u)$  have no common roots.

Traveling wave solutions of (1) correspond to bounded orbits of (3) different from singular points. Due to the structure of system (3) it is impossible to have a homoclinic orbit or a limit cycle in the phase plane of (3), which yields that it is necessary to have at least two singular points of (3) and a heteroclinic orbit connecting them (see Fig. 1a,b) to prove existence of traveling wave solutions of (1) satisfying (C1), (C2).

In general, smooth functions  $h(u)$ ,  $g(u, v)$  can be written in the form

$$h(u) = \tilde{h}(u)(u - \hat{u}_1) \dots (u - \hat{u}_m), \quad \tilde{h}(u) \neq 0 \text{ for any } u,$$

$$g(u, v) = g_1(u)\tilde{g}_2(v)(v - \hat{v}_1) \dots (v - \hat{v}_n), \quad \tilde{g}_2(v) \neq 0 \text{ for any } v.$$

The following proposition holds for neighboring roots of  $h(u)$  and  $g_2(v)$ .

### Proposition 2.

- i. *Let the wave system (3) satisfying (A1)-(A3) have singular points  $(\hat{u}, \hat{v}_1)$  and  $(\hat{u}, \hat{v}_2)$ , where  $\hat{v}_1$  and  $\hat{v}_2$  are neighboring roots of  $g_2(v)$  then one of these points is a saddle and the other one is a node.*
- ii. *Let the wave system (3) satisfying (A1)-(A3) have singular points  $(\hat{u}_1, \hat{v})$  and  $(\hat{u}_2, \hat{v})$ , where  $\hat{u}_1$  and  $\hat{u}_2$  are neighboring roots of  $h(u)$ , then these points can both be saddles, nodes or one is a node and another is a saddle.*

*Proof.* Let  $(\hat{u}, \hat{v})$  be a singular point of (3). The eigenvalues of this point  $\lambda_1(\hat{u}, \hat{v}) = h'(\hat{u})$  and  $\lambda_2(\hat{u}, \hat{v}) = g_1(\hat{u})g_2'(\hat{v})$  are real numbers (henceforth we use prime to denote differentiation when it is clear with respect to which variable it is carried out). This implies that singular point  $(\hat{u}, \hat{v})$  of system (3) cannot be a focus or center.

- i. The claim is a simple conjecture of condition (A2).
- ii. Let us consider two equilibrium points  $(\hat{u}_1, \hat{v})$  and  $(\hat{u}_2, \hat{v})$ . The eigenvalues  $\lambda_1(\hat{u}_1, \hat{v})$  and  $\lambda_1(\hat{u}_2, \hat{v})$  have opposite signs due to (A1).

Consider another pair of eigenvalues  $\lambda_2(\hat{u}_1, \hat{v})$  and  $\lambda_2(\hat{u}_2, \hat{v})$  and assume that (A3) holds. If the number of roots of  $g_1(u)$  located between  $\hat{u}_1$  and  $\hat{u}_2$  is even (or zero) then the signs of these eigenvalues are the same. This implies that one of the equilibrium points is a saddle whereas the other one is a node. If the number of roots of  $g_1(u)$  located between  $\hat{u}_1$  and  $\hat{u}_2$  is odd then the signs of these eigenvalues are opposite which implies that both equilibria are saddles or nodes (one node is attracting and another is repelling).  $\square$

Note, that in case ii of Proposition 2 in order to guarantee that both singular points are nodes one should have  $h'(\hat{u}_1)g_1(\hat{u}_1)g_2'(\hat{v}) > 0$  and  $h'(\hat{u}_2)g_1(\hat{u}_2)g_2'(\hat{v}) > 0$ . Due to continuity

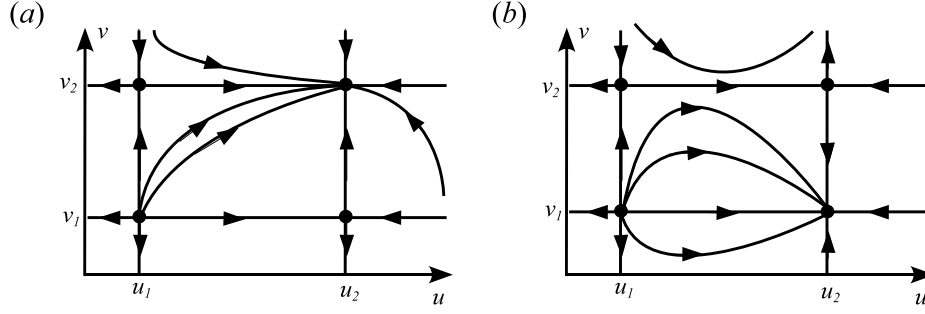


Figure 2: Two types of orbit cells of system (3)

arguments there exists a family of orbits of (3) which tend to one of the nodes when  $z \rightarrow \infty$  and to the other node when  $z \rightarrow -\infty$ .

Taking into account that straight lines  $u = \hat{u}_i$ ,  $i = 1..m$  and  $v = \hat{v}_j$ ,  $j = 1..n$  consist of orbits of system (3) we obtain that the phase plane of (3) is divided into  $(m-1)(n-1)$  bounded rectangular domains whose boundaries are  $u = \hat{u}_i$ ,  $i = 1..m$  and  $v = \hat{v}_j$ ,  $j = 1..n$ . We shall call these domains *the orbit cells*.

Due to Proposition 2 it immediately follows that an orbit cell can be one of the following two types (up to  $\pi$ -degree rotation) that are presented in Fig. 2. The behavior of the orbits inside a cell is completely described by the types of the singular points at the corners of the cell. Moreover, any orbit inside a cell corresponds to a bounded traveling wave solution of system (1).

Summarizing the previous analysis we obtain the following theorem.

**Theorem 1.** *The system (1) satisfying (C1), (C2) and (A1)-(A3) possesses traveling wave solutions*

- i. *of a front-front type (Fig. 1a) if and only if the wave system (3) has four singular points  $(u_i, v_i)$ ,  $i = 1, 2$ , which are the vertexes of a bounded orbit cell and every two neighboring vertexes are a node and a saddle (Fig. 2a);*
- ii. *of a front-impulse type (Fig. 1b) if and only if the wave system (3) has two neighboring nodes  $(\hat{u}_1, \hat{v})$  and  $(\hat{u}_2, \hat{v})$ , (see Fig. 2b).*

**Remarks to Theorem 1.**

1. In both cases the orbits of system (3) that correspond to traveling wave solutions of (1) are dense in the corresponding orbit cell.

2. System (1) has a traveling wave solution which is a front in  $v$ -component and space-homogeneous in  $u$ -component if and only if the wave system (3) has neighboring saddle and node with identical  $u$ -coordinate, see singular points  $(u_1, v_1), (u_1, v_2)$  and  $(u_2, v_1), (u_2, v_2)$  in Fig. 2b.

It is possible to write down asymptotics for  $u$  and  $v$  profiles (these asymptotics can be used, e.g., as initial conditions for numerical solutions of (1)). We present these asymptotics only in the simplest case.

Let us assume that the wave system has the form

$$\begin{aligned} u' &= A(u - \hat{u}_1)(u - \hat{u}_2), \\ v' &= -B(v - \hat{v}_1)(v - \hat{v}_2)(u - \tilde{u}), \end{aligned} \quad (4)$$

where  $A, B > 0$  are constant. An explicit solution of (4) is

$$\begin{aligned} u(z) &= \hat{u}_2 + \frac{\hat{u}_1 - \hat{u}_2}{1 + C_1 \exp \{A(\hat{u}_1 - \hat{u}_2)z\}}, \\ v(z) &= \hat{v}_2 + \frac{(\hat{v}_1 - \hat{v}_2)(1 + C_1 \exp \{A(\hat{u}_1 - \hat{u}_2)z\})^{\frac{B(\hat{v}_2 - \hat{v}_1)}{A}}}{(1 + C_1 \exp \{A(\hat{u}_1 - \hat{u}_2)z\})^{\frac{B(\hat{v}_2 - \hat{v}_1)}{A}} + C_2 \exp \{B(\tilde{u} - \hat{u}_1)(\hat{v}_1 - \hat{v}_2)z\}}, \end{aligned} \quad (5)$$

where  $C_1, C_2$  are arbitrary constants. We emphasize here that even with fixed  $c$  and  $\alpha$  there is a two-parameter family of wave profiles.

Let  $\hat{u}_1 < \tilde{u} < \hat{u}_2$  and  $\hat{v}_1 < \hat{v}_2$ . We consider non-trivial profiles ( $C_1 C_2 \neq 0$ ). It is straightforward to show that  $u(-\infty) = \hat{u}_2, u(\infty) = \hat{u}_1$  and, hence,  $u$ -profile is a front;  $v(-\infty) = \hat{v}_1, v(\infty) = \hat{v}_2$ , and  $v$ -profile is an impulse. If  $\tilde{u} < \hat{u}_1$  or  $\tilde{u} > \hat{u}_2$  then  $u$ -profile remains the same and  $v$ -profile becomes a front.

Formulas (5) can be used as a first approximation for wave profiles of system (1) even in the case  $A = A(u), B = B(u, v)$  where  $A(u)$  and  $B(u, v)$  do not change the sign when  $u \in (\hat{u}_1, \hat{u}_2), v \in (\hat{v}_1, \hat{v}_2)$ .

It is worth noting that if  $\hat{u}_1 = \tilde{u}$  then

$$v(\infty) = \hat{v}_2 + \frac{\hat{v}_1 - \hat{v}_2}{1 + C_2},$$

i.e., the boundary of the profile depends on an arbitrary constant; we deal with such solutions in the next section.

## 2.2 Bifurcations of the travelling wave solutions

Considering  $c$  and  $\alpha$  as bifurcation parameters we can relax some of non-degeneracy conditions (A1)-(A3).

First we note that the right-hand side of the second equation of system (3) does not depend in a non-trivial way on  $c$  and  $\alpha$  and we will not consider the case when (A2) is violated. In general, by varying the bifurcation parameters we can only achieve that either (A1) or (A3) do not hold. We shall show that in the latter case new traveling wave solutions can appear in system (1) satisfying (C1), (C2).

First let us assume that (A1) does not hold, i.e., function  $h(u)$  has a root  $\hat{u}$  of multiplicity  $m > 1$  for some  $\alpha^*, c^*$ , and the wave system (3) has a complicated singular point  $(\hat{u}, \hat{v})$ . The system can be written in the form

$$\begin{aligned} u' &= (u - \hat{u})^m q_1(u), \\ v' &= (v - \hat{v}) g_1(u) q_2(v), \end{aligned} \quad (6)$$

where  $q_1(\hat{u}) q_2(\hat{v}) g_1(\hat{u}) \neq 0$ . Then the singular point  $(\hat{u}, \hat{v})$  of system (6) is either a saddle, a node, or a saddle-node [14]. For the first two types of critical points, the structure of the phase

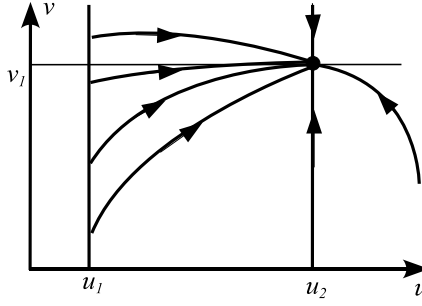


Figure 3: The phase plane of system (3);  $u_1$  is a root of both  $h(u)$  and  $g_1(u)$ . The wave solutions corresponding to bounded orbits of the wave system form a free-boundary family

plane of the wave system was completely described above. In case of a saddle-node the line  $u = \hat{u}$  divides the plane such that in one half-plane the singular point is topologically equivalent to a node, and in the other one it is topologically equivalent to a saddle; due to the fact that  $u = \hat{u}$  consists of solutions of (3) this type of singular points does not yield qualitatively new bounded solutions of (3). Therefore, violation of (A1) does not result in new types of wave solutions of (1) satisfying (C1), (C2).

Appearance or disappearance of  $u$ -fronts correspond to appearance or disappearance of the roots of the function  $h(u)$  which can occur with variation of the parameters  $c$  and  $\alpha$ . The simplest case of the appearance of two or three roots corresponds to the fold or cusp bifurcations respectively [15] in the first equation of (3). The simple conditions for the fold and cusp bifurcations show that, under variation of the boundary conditions (parameter  $\alpha$ ) and the wave speed (parameter  $c$ ), appearance of traveling wave solutions of (1) is possible.

Now we assume that (A3) does not hold, i.e., the functions  $g_1(u)$  and  $h(u)$  have a coinciding root  $\hat{u}$ . In this case system (3) has a line of non-isolated singularities in the phase plane  $(u, v)$ . Each point of the form  $(\hat{u}, v)$  is a non-isolated singular point; all the points on the line  $u = \hat{u}$  are either simultaneously attracting or repelling in a transversal direction to this line [14]. If we assume that there exists a node  $(\hat{u}_1, \hat{v})$  of (3) such that  $\hat{u}_1$  is a root of  $h(u)$ ,  $g_1(\hat{u}_1) \neq 0$ , and there are no other roots of  $h(u)$  between  $\hat{u}$  and  $\hat{u}_1$  then, due to continuity arguments, there exists a family of bounded orbits of (3) (Fig. 3). To describe the traveling wave solutions corresponding to this family we define

**Definition 2.** We shall say that model (1) possesses a family of free-boundary wave fronts in  $v$ -component if a) every  $v(z) \rightarrow \hat{v}$  when  $z \rightarrow \infty$  ( $z \rightarrow -\infty$ ); b) there exists an interval  $(v_1, v_2)$  such that for any  $v^* \in (v_1, v_2)$  it is possible to find  $v$ -profile with the property  $v(z) \rightarrow v^*$  when  $z \rightarrow -\infty$  ( $z \rightarrow \infty$ ).

Summarizing we obtain

**Proposition 3.** The system (1) satisfying (C1), (C2), (A1), (A2) has a traveling wave solution such that  $u$ -profile is a front and  $v$ -profile is a free boundary front if and only if condition (A3) is violated and there is a node of system (3) such that there are no other singular points of (3) between this node and the line of non-isolated singular points.



The primary importance of such traveling wave solutions comes from the fact that for an arbitrary boundary condition (from a particular interval) for system (1) we can find a wave solution whose  $v$ -profile is a front. Note that violation of (A3) and simultaneous appearance of a free-boundary  $v$  component naturally occurs when the roots of  $h(u)$  are shifted under variation of  $c$  and  $\alpha$ . It follows from (3) that bifurcation of  $v$ -profile occurs at  $(c^*, \alpha^*)$  such that  $\hat{u} = -\alpha^*/c^*$  is a simple root of  $g_1(u)$ .

### 2.3 An illustrative example

Here we present a simple example to illustrate the theoretical analysis from the previous sections.

We consider model (1) with

$$f(u, v) = \frac{(u-l)(1-u)}{v(v-1)}, \quad g(u, v) = -k(u-r)v(v-1), \quad (7)$$

where  $l, k, r$  are non-negative parameters. The particular form of the functions  $f(u, v), g(u, v)$  obviously satisfies (C1) and (C2).

The wave system reads

$$\begin{aligned} u' &= cu - k(u-l)(1-u)(u-r) + \alpha, \\ v' &= -k(u-r)v(v-1)/c. \end{aligned} \quad (8)$$

The system (8) can have up to six singular points. For instance, if we fix the parameter values  $l = 0.2, r = 0.4, k = 1, \alpha = -0.1, c = 0.3$  then system (8) possesses six singular points; therefore, there are two orbit cells ensuring existence of traveling wave solutions of system (1).

A phase portrait of (8) is shown in Fig. 4. From Fig. 4 it can be seen that, with the given parameter values, there exist two qualitatively different traveling wave solutions of the initial cross-diffusion system which correspond to two cases of Theorem 1.

Numerical solutions of system (1) with functions (7) and the given parameter values are shown in Fig. 5 (the details of the numerical computations are presented in the Appendix).

If we change the value of  $\alpha$  to  $-0.12$  then we obtain a family of free-boundary traveling wave solutions.

## 3 Wave solutions of the separable models and some generalizations

### 3.1 General theory

In this section we study models (1) which satisfy (C1). The rational function  $f_2(v) = Z(v)/R(v)$  can be presented in the form

$$f_2(v) = \frac{Z(v)}{R(v)} = \frac{\tilde{Z}(v)(v-v_1)\dots(v-v_m)}{\tilde{R}(v)(v-\check{v}_1)\dots(v-\check{v}_k)},$$

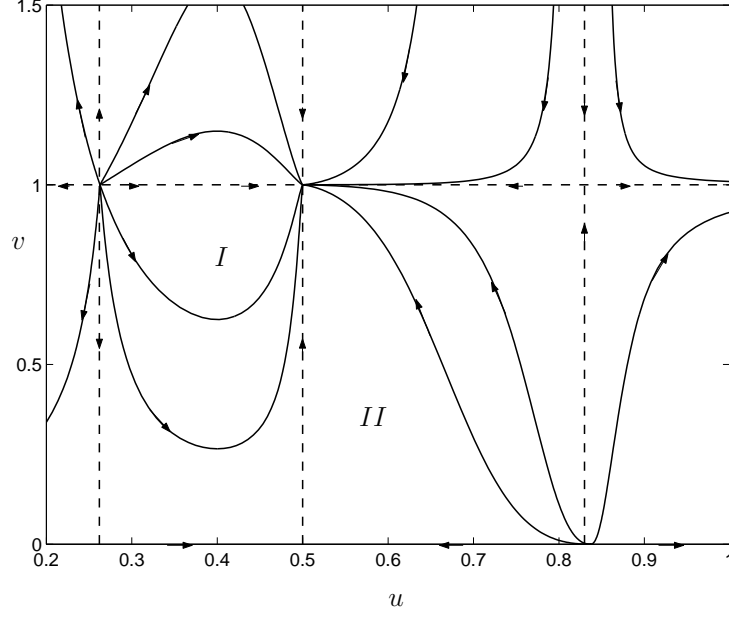


Figure 4: The phase portrait of system (8). The parameters are  $l = 0.2$ ,  $r = 0.4$ ,  $k = 1$ ,  $\alpha = -0.1$ ,  $c = 0.3$ .

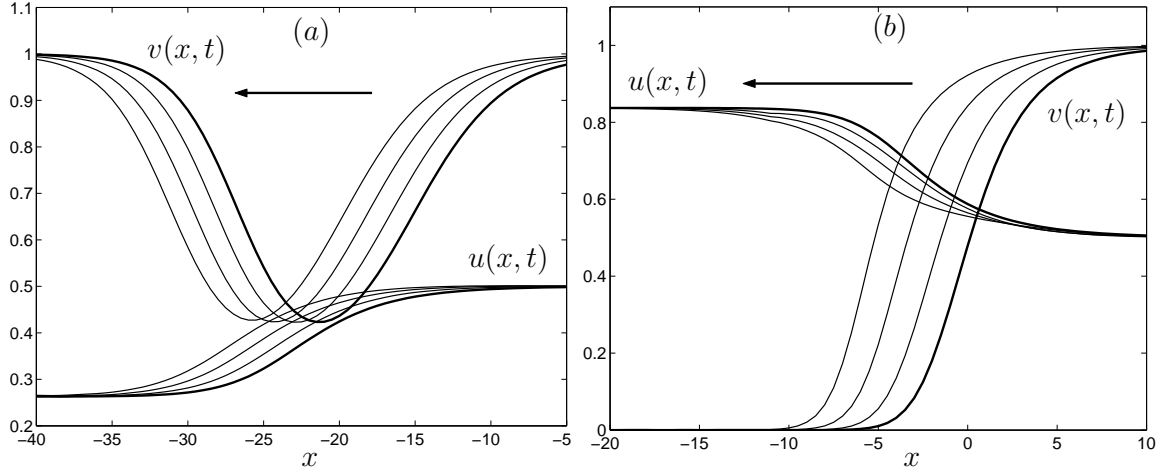


Figure 5: Numerical solutions of system (1) with functions (7). The initial conditions are chosen to start the calculations from the orbit cells labelled *I* (panel (a), front-impulse solution) or *II* (panel (b), front-front solution) in Fig. 4. The solutions are shown for the time moments  $t_0 = 0$  (bold curves)  $< t_1 < t_2 < t_3 = 18$  in equal time intervals

where  $\tilde{Z}(v), \tilde{R}(v)$  do not have real roots;  $m \geq 0, k > 0$ ;  $v_i \neq v_j$  for any  $i, j$ . The wave system has the form

$$\begin{aligned} u' &= cu + \alpha + f_1(u)g_1(u)g_2(v)\frac{Z(v)}{cR(v)}, \\ v' &= g_1(u)g_2(v)/c. \end{aligned} \tag{9}$$

By transformation of the independent variable

$$dy = \frac{dz}{cR(v)}, \quad (10)$$

which is smooth for any  $v$  except for  $v = \check{v}_i$ ,  $i = 1..k$ , system (9) becomes

$$\begin{aligned} \frac{du}{dy} &= c^2 R(v)(u + \alpha/c) + f_1(u)g_1(u)g_2(v)Z(v), \\ \frac{dv}{dy} &= g_1(u)g_2(v)R(v). \end{aligned} \quad (11)$$

The roots of functions  $f_1(u), g_1(u), g_2(v), Z(v), R(v)$  do not depend on parameters  $c$  and  $\alpha$ , hence, we will suppose that the following conditions of non-degeneracy are fulfilled:

- (B1)  $R(v)$  and  $g_2(v)$  have no common roots;
- (B2)  $f_1(u)$  and  $g_1(u)$  have no common roots;
- (B3)  $f_1(u), g_1(u), g_2(v), R(v)$  have no multiple roots.

Coordinates of singular points  $(u, v) = (\hat{u}, \hat{v})$  of (11) can be found from one of the systems:

$$-\alpha/c = u, \quad g_2(v) = 0, \quad (12)$$

$$R(v) = 0, \quad f_1(u) = 0, \quad (13)$$

$$R(v) = 0, \quad g_1(u) = 0, \quad (14)$$

or from combination of (12)-(14).

To infer possible types of the singular points of (11) we consider  $D = \text{tr}^2 J - 4 \det J$ , where  $J$  is the Jacobian of (11) evaluated at a singular point  $(\hat{u}, \hat{v})$ .

If  $(\hat{u}, \hat{v})$  is a solution of (12) then

$$\begin{aligned} \text{tr } J &= R(\hat{v})(c^2 + g_1(\hat{u})g_2'(\hat{v})), \\ \det J &= c^2 g_1(\hat{u})R(\hat{v})^2 g_2'(\hat{v}), \\ D &= R(\hat{v})^2 (c^2 - g_1(\hat{u})g_2'(\hat{v}))^2; \end{aligned}$$

if  $(\hat{u}, \hat{v})$  is a solution of (13) then

$$\begin{aligned} \text{tr } J &= g_1(\hat{u})g_2(\hat{v})(Z(\hat{v})f_1'(\hat{u}) + R'(\hat{v})), \\ \det J &= (g_1(\hat{u})g_2(\hat{v}))^2 Z(\hat{v})f_1'(\hat{u})R'(\hat{v}), \\ D &= (g_1(\hat{u})g_2(\hat{v}))^2 (Z(\hat{v})f_1'(\hat{u}) - R'(\hat{v}))^2; \end{aligned}$$

if  $(\hat{u}, \hat{v})$  is a solution of (14) then

$$\det J = 0, \quad \text{tr } J = f_1(\hat{u})g_2(\hat{v})Z(\hat{v})g_1'(\hat{u}).$$

Consequently we obtain that  $(\hat{u}, \hat{v})$  is a saddle or a node for the cases corresponding to (12) and (13), and is a saddle, node, or saddle-node (see [14]) in the case (14). Just as for the reduced separable models there are no singular points of (11) of center or focus type.

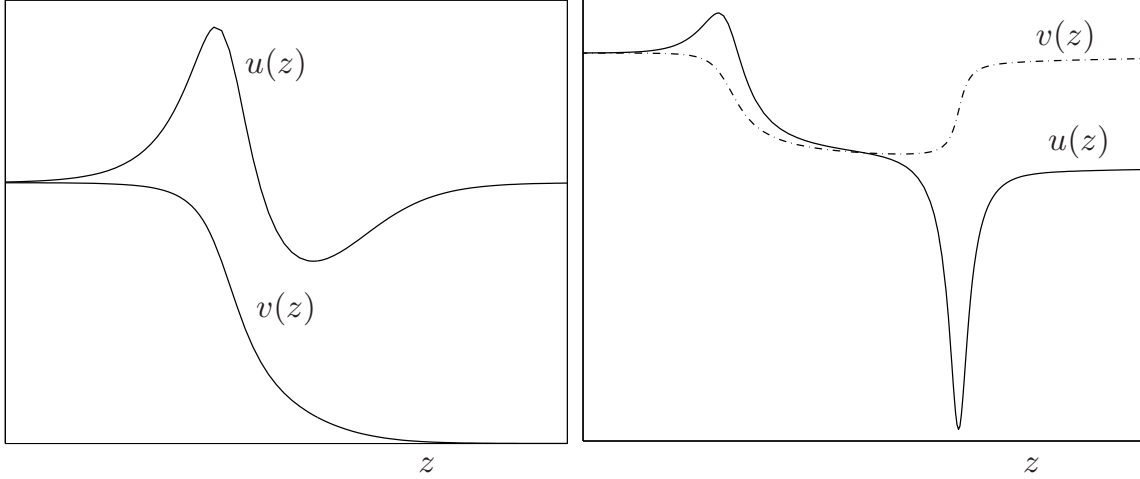


Figure 6: Complex shape of wave profiles of model (9). The functions are  $f_1(u) = u - 1$ ,  $g_1(u) = u$ ,  $g_2(v) = 1$ ,  $R(v) = v(v - 1)$ ,  $Z(v) = 2v - 1$ . The left panel shows an impulse-front solution, the right panel shows a front-impulse solution

Here we do not pursue the problem of classification of possible structures of the phase plane of (11) and are only concerned with new types of traveling wave solutions. Analyzing formulas for  $\det J$  and  $\text{tr } J$  and applying arguments in the line with the proof of Proposition 2 and Theorem 1 we obtain

**Proposition 4.** *Let coordinates of singular points of (11) satisfy (12) and function  $g_2(v)$  have two real neighboring roots  $\hat{v}_1$  and  $\hat{v}_2$ . If the function  $R(v)$  has an odd number of roots between  $\hat{v}_1$  and  $\hat{v}_2$  and point  $(\hat{u} = -\alpha/c, \hat{v}_1)$  is a node, then  $(\hat{u} = -\alpha/c, \hat{v}_2)$  is a node as well (and vice versa).*

*Let coordinates of singular points of (11) satisfy (13) and  $u = \hat{u}$  be a root of  $f_1(u)$ ,  $R(v)$  have two real neighboring roots  $\hat{v}_1$  and  $\hat{v}_2$ . If the function  $Z(v)$  has an odd number of roots between  $\hat{v}_1$  and  $\hat{v}_2$  and point  $(\hat{u}, \hat{v}_1)$  is a node, then  $(\hat{u}, \hat{v}_2)$  is a node as well (and vice versa).*

Due to the structure of system (11) the phase plane is divided into horizontal strips, whose boundaries are given by  $v = \hat{v}$ , where  $\hat{v}$  is a root of  $R(v)$  or  $g_2(v)$ ; all singular points of (11) are situated on these boundaries. Bringing in the continuity arguments we obtain that under the conditions of Proposition 4 there is a family of bounded orbits of (11) which correspond to the traveling wave solution of (1) of an impulse-front type.

It is worth noting that the structure of the phase plane inside a strip can be quite arbitrary, and we can only indicate asymptotic behavior of orbits in neighborhoods of singular points. As a result, under given boundary conditions (or, equivalently, fixed  $\alpha$ ) families of traveling wave solutions may have complex shapes (opposite to the examples presented in Fig. 1). For instance, there can be non-monotonous fronts with humps or impulses which also have multiple humps and hollows. In general, we can only state that the form of impulses and fronts can be quite arbitrary, which is illustrated in Fig. 6.

Under variation of parameters  $c$  and  $\alpha$  it is possible that function  $g_1(u)$  has a root  $\hat{u} = -\alpha^*/c^*$

for some values of the parameters; in this case system (11) has a line of non-isolated singular points  $u = \hat{u}$  and the analysis in this situation is similar to the analysis which led to Proposition 3: a family of free-boundary fronts appears in  $v$  component.

Now let us relax the condition (B2); we assume that there exists such  $\hat{u}$  that  $f_1(\hat{u}) = 0$  and  $g_1(\hat{u}) = 0$ . We can always find values of  $c$  and  $\alpha$  such that  $\hat{u} = -\alpha/c$ . In this case the phase plane of (11) has a line  $u = \hat{u}$  of non-isolated singular points. After the change of the independent variable  $d\tau = dy/(u - \hat{u})$  the resulting system still possesses singular point of the form  $(\hat{u}, \hat{v})$ , where  $\hat{v}$  is a root of  $R(v)$ . If this point is a node then, applying continuity arguments, we obtain that there exist a family of orbits of system (11) such that some solutions from a neighborhood of  $(\hat{u}, \hat{v})$  tend to this point if  $y \rightarrow \infty$  (or  $y \rightarrow -\infty$ ) and tend to point of the form  $(\hat{u}, v^*)$  if  $y \rightarrow -\infty$  (or  $y \rightarrow \infty$ ), where  $v^*$  is an arbitrary constant from some interval. These solutions correspond to traveling wave solution of (1) such that  $u$ -profile is an impulse and  $v$  component is a free-boundary front (see Fig. 8).

Summarizing we obtain

**Theorem 2.** *The system (1) satisfying (C1) and (B1)-(B3) can only possess traveling wave solutions of the following kinds:*

- i. *front-front solutions;*
- ii. *front-impulse solutions;*
- iii. *impulse-front solutions;*
- iv. *Under variation of parameters  $c$  and  $\alpha$  it is possible to have wave solutions where  $u$  component is a front,  $v$  component is a free-boundary front;*
- v. *Under the additional condition that (B2) does not hold it is possible to have wave solutions with  $u$  component is an impulse and  $v$  component is a free-boundary front.*

### 3.2 The phase plane analysis of the Keller-Segel model

The classical Keller-Segel model has the form (1) with  $f(u, v) = \delta u/v$ ,  $g(u, v) = -ku$ , where  $\delta, k > 0$  (see [3]). In our terminology this model falls in the class of the separable models; the wave system reads

$$\begin{aligned} u' &= cu - \delta k \frac{u^2}{cv} + \alpha, \\ v' &= -ku/c, \end{aligned} \tag{15}$$

which, with the help of transformation (10), can be reduced to the form (11):

$$\begin{aligned} \frac{du}{dy} &= c^2 uv - \delta k u^2 + \alpha cv, \\ \frac{dv}{dy} &= -kuv. \end{aligned} \tag{16}$$

If  $\alpha \neq 0$  then system (16) has the only singular point  $(u, v) = (0, 0)$ . In this case the separable model cannot possess traveling wave solutions (Theorem 2). Hence we put  $\alpha = 0$ . Note that the last requirement is necessary if one supposes that  $U(x, t)$  should be finite.

For  $\alpha = 0$  system (16) has a line of non-isolated singular points  $(0, v)$ , and there is also additional degeneracy at the point  $(0, 0)$  (case v. in Theorem (2)). This can be seen applying the second transformation of the independent variable  $d\tau = udy$ , which leads to the system

$$\begin{aligned}\frac{du}{d\tau} &= c^2v - \delta ku, \\ \frac{dv}{d\tau} &= -kv,\end{aligned}\tag{17}$$

for which the origin is a topological node with the eigenvalues  $\lambda_1 = -k\delta$  and  $\lambda_2 = -k$ . Thus there exists a family of traveling wave solutions whose  $u$ -profile is an impulse and  $v$ -profile is a free-boundary front.

In Fig. 7 we show how the parametrization of the phase curves of the wave system change after the transformations of the independent variables. This picture can also serve as an illustration to assertion v. of Theorem 2.

Due to biological interpretation of the Keller-Segel model it is necessary to have  $U(x, t) \geq 0$  and  $V(x, t) \geq 0$ . Using the fact that an eigenvector  $(1, (\delta - 1)k/c^2)$  corresponds to  $\lambda_1$  and  $(1, 0)$  corresponds to  $\lambda_2$  it is straightforward to see that to ensure the existence of non-negative travelling way solutions we should have  $\delta > 1$ .

Numerical solutions of the Keller-Segel model are given in Fig. 8b. Originally, the Keller-Segel model was suggested to describe movement of bands of *E. coli* which were observed to travel at a constant speed when the bacteria are placed in one end of a capillary tube containing oxygen and an energy source [3]. In Fig. 8b it can be seen that bacteria ( $u(x, t)$ ) seek an optimal environment: the bacteria avoid low concentrations and move preferentially toward higher concentrations of some critical substrate ( $v(x, t)$ ). The stability of the traveling solutions found was studied analytically in [19, 21].

### 3.3 Impulse-impulse solutions

In the preceding sections we studied the systems (1) where the functions  $f(u, v), g(u, v)$  can be represented as a product of functions that only depend on one variable. The next natural step

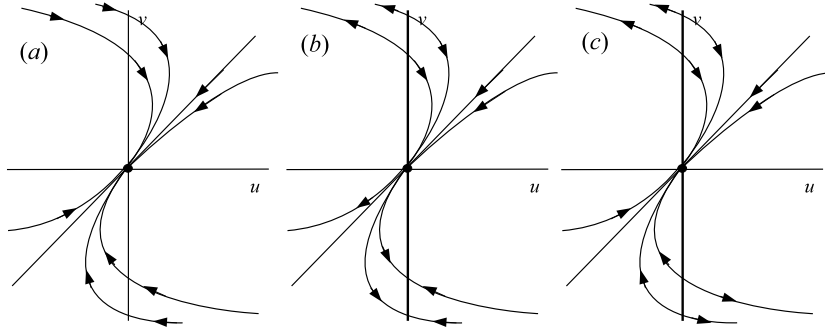


Figure 7: The phase planes of systems (17) (a), (16) (b) and (15) (c)

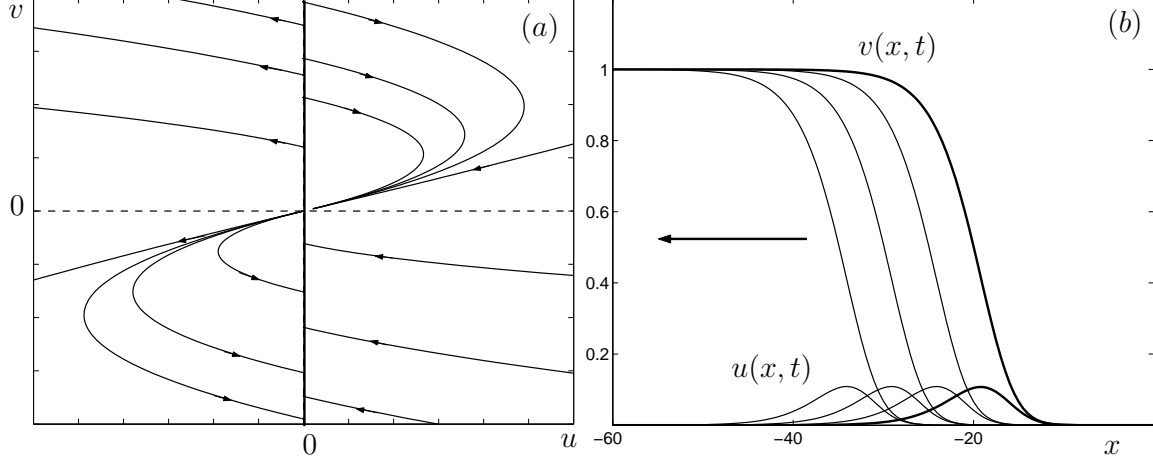


Figure 8: (a) The phase plane of system (15); the parameters are  $\alpha = 0, k = 1, \delta = 4, c = 0.5$ . (b) Numerical solutions of the Keller-Segel system with the parameters given in (a). The solutions are shown for the time moments  $t_0 = 0$  (bold curves)  $< t_1 < t_2 < t_3 = 30$  in equal time intervals

is to assume that these functions depend on affine expressions  $au + bv + c$ , where  $a, b, c$  are not equal to zero simultaneously. Here we present an explicit example of such a system. The example is motivated by appearance of a particular type of traveling wave solutions, which is absent in the separable models.

We suppose that

$$\begin{aligned} f(u, v) &= \frac{\delta u}{\beta u + v}, \quad \delta > 0, \beta \geq 0, \\ g(u, v) &= -ku + rv, \quad k, r \geq 0. \end{aligned} \quad (18)$$

If  $\beta, r = 0$  then we have the Keller-Segel model studied in Section 3.2.

The wave system for (1) with the functions given by (18) reads

$$\begin{aligned} u' &= cu + \frac{\delta u(-ku + rv)}{c(\beta u + v)}, \\ v' &= (-ku + rv)/c, \end{aligned} \quad (19)$$

where we put parameter  $\alpha$  equal zero.

After the change of the independent variable  $dz/(c(\beta u + v)) = dy$  system (19) takes the form

$$\begin{aligned} \frac{du}{dy} &= c^2 u(\beta u + v) + \delta u(-ku + rv), \\ \frac{dv}{dy} &= (-ku + rv)(\beta u + v). \end{aligned} \quad (20)$$

If  $r = 0$  then system (20) has a line on non-isolated singular points  $u = 0$ ; if  $r \neq 0$  then  $(u, v) = (0, 0)$  is an isolated non-hyperbolic singular point of (20) (i.e., both eigenvalues of the Jacobian evaluated at this point are zero).

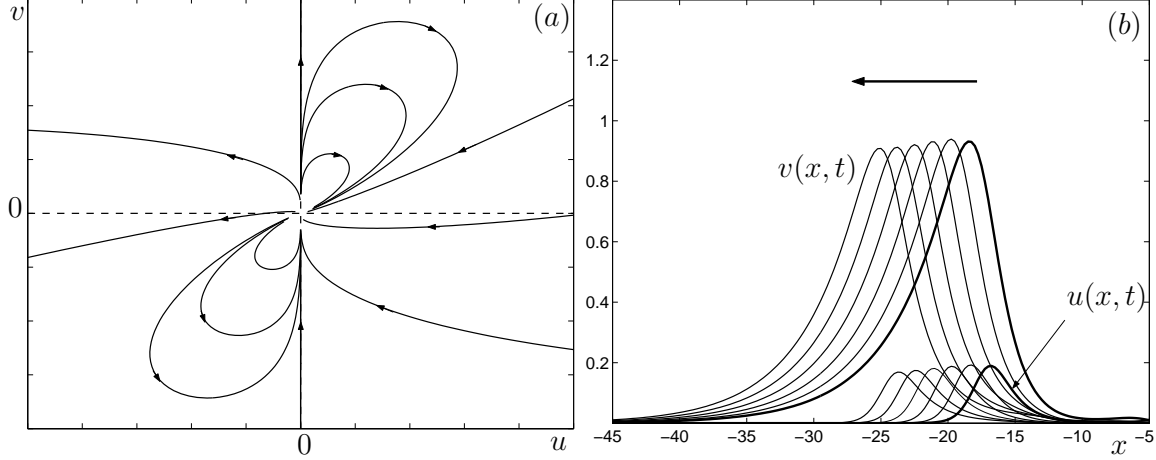


Figure 9: (a) The phase plane of system (19); the parameters are  $k = 1, \beta = 1, \delta = 2, c = 0.43, r = 0.1$ . (b) Numerical solutions of system (1) with the functions given by (18) and the parameter values as in (a). The solutions are shown for the time moments  $t_0 = 0$  (bold curves)  $< t_1 < t_2 < t_3 < t_4 < t_5 = 20$  in equal time intervals

First we consider the case  $r = 0$ . After yet another transformation  $d\tau = dy/u$  we obtain

$$\begin{aligned} \frac{du}{d\tau} &= (c^2\beta - \delta k)u + c^2v, \\ \frac{dv}{d\tau} &= -k(\beta u + v). \end{aligned} \quad (21)$$

Thus for  $\delta > 1$  and  $0 \leq \beta \leq k(\sqrt{\delta} - 1)^2/c^2$  the origin is a node for system (21) which implies that in the initial system (20) there exist a family of bounded orbits which tend to  $(0, 0)$  for  $y \rightarrow \infty$  or  $y \rightarrow -\infty$ . This family corresponds to a family of traveling wave solution of the system (1) where  $u$ -profile is an impulse and  $v$ -profile is a free-boundary front. The picture is topologically equivalent to the phase portrait shown in Fig. 8a.

For  $r \neq 0$  the wave system (20) has singular point  $(0, 0)$  possessing two elliptic sectors in its neighborhood (see Fig. 9a). The proof of existence of the elliptic sectors can easily be conducted with the methods given in [24]. Asymptotics of homoclinics composing the elliptic sector are  $u = 0$  (trivial) and  $v = K^+u$ , where  $K^+$  is the biggest root of the equation

$$K^2(c^2 + r(\delta - 1)) + K(\beta c^2 - k(\delta - 1) - \beta r) + \beta k = 0.$$

The family of homoclinics in the phase plane  $(u, v)$  correspond to the family of wave impulses for the system (1) (see Fig. 1c and Fig. 9b). To our knowledge such kind of solutions (infinitely many traveling wave solutions of impulse-impulse type with the fixed values of the parameters) was not previously described in the literature.

The results of the numerical computations (Fig. 9b) indicate that the family of traveling impulses is clearly non-stationary, since its amplitude decays visibly in time. Which is important, however, is that it is possible to observe moving impulses at least at a finite time interval.



## 4 Conclusions

In this paper we described all possible traveling wave solutions  $\{u(z), v(z), z = x + ct\}$  of the cross-diffusion two-component PDE model (1) satisfying (C1), where the cross-diffusion coefficient may depend on both variables and possess singularities. Such kind of models is widely used in modeling populations that can chemotactically react to an immovable signal (attractant) (e.g., [1, 3] and references therein). The study of traveling wave solutions of model (1) was carried out by qualitative and bifurcation analysis of the phase portraits of the wave system (2) that depends on parameters  $c$  and  $\alpha$ . Here  $c$  is a speed of wave propagation and  $\alpha$  characterizes boundary conditions under given  $c$ . Any traveling wave solution of (1) with given boundary conditions corresponds to a solution of wave system (2) with specific values of parameters  $c$  and  $\alpha$ ; the converse is also true. Therefore, instead of trying to construct a traveling wave solution with the given boundary conditions, we study the set of all possible bounded solutions of the wave system considering  $c$  and  $\alpha$  as its parameters. This approach allows identifying all boundary conditions for which model (1) possesses traveling wave solutions.

The main attention is paid to the so-called separable model, i.e., to model (1) that satisfies (C1); it is worth noting that in this case the functions  $f(u, v)$  and  $g(u, v)$  in (1) are products of factors that depend on a single variable. We showed that for some fixed values of parameters  $c$  and  $\alpha$  the solutions of the wave system compose two-parameter family (for explicit formulas in a simple particular case see (5)). One result is that  $\{u(z), v(z)\}$ -profiles of the wave solution of the reduced separable models (1) that satisfies (C1), (C2) can be only front-front or front-impulse; for more general case of the separable models we can additionally have impulse-front solutions. For some special relations between  $c$ ,  $\alpha$  and the model parameters model (1) can have wave solutions whose  $v$ -profiles are free-boundary fronts, i.e.,  $v(z)$  tends to an arbitrary constant from some interval at  $z \rightarrow \infty$  or at  $z \rightarrow -\infty$ . Note that traveling wave solutions of the well-known Keller-Segel model [3] as well as of its generalization (Section 3.3) have impulse-front profiles with a free-boundary front (Sections 3.2, 3.3).

We also considered a natural extension of the separable models; namely, we gave an example of model (1) where  $f, g$  are products of factors that depend on affine expression of both variables (see (18)). This model can be considered as a generalization of the Keller-Segel model because it has two additional parameters and turns into the Keller-Segel model if both of these parameters are zero. If only one of the parameters vanishes, the model has a family of “Keller-Segel”-type solutions, i.e.,  $u$ -profile is an impulse and  $v$ -profile is a frond with a free boundary. Importantly, in some parameter domains this model possesses a two-parameter family of impulse-impulse solutions (Fig. 9). To the best of our knowledge, such type of traveling wave solutions was not previously described in the literature: depending on the initial conditions traveling impulses can have quite a different form for the fixed values of the system parameters. Taking into account the fact that such solutions are absent in the separable models, we can consider model (1) with functions (18) as the simplest model possessing this type of traveling wave solutions.

Rearrangements of traveling wave solutions of PDE model (1), which occur with changes of the wave propagation velocity and the boundary conditions, correspond to bifurcations of its ODE wave system. In particular, appearance/disappearance of front-profiles with variation of parameters  $c$  and  $\alpha$  correspond to the fold or cusp bifurcations in the wave system; rearrangement of a front to an impulse can be accompanied by appearance/disappearance of non-isolated

singular points in the phase plane of the corresponding wave system (see Section 2.3). Existence of non-isolated singular points in the wave system may result in the existence of free-boundary fronts in model (1). For instance, this is the case for the Keller-Segel model.

We emphasize that the separable model (1), when the values of parameters  $c$  and  $\alpha$  fixed in the wave system, possesses, in general, two-parameter family of traveling wave solutions. There are infinitely many bounded orbits of (2) that correspond to traveling wave solutions of (1) (see Figures 2, 5, 8, 9). It is of particular interest that in all numerical solutions of (1) that we conducted it is possible to observe traveling waves. We did not discuss the issue of stability of the traveling wave solutions found, but we note that it is usually true that unstable solutions cannot be produced in numerical calculations. It is tempting to put forward a hypothesis that the presence of additional degrees of freedom (two free parameters) is the reason of producing traveling waves in numerical computations. This important question can be a subject of future research.

## 5 Appendix

We did numerical simulations of system (1) for  $x \in [-L, L]$ , where  $L$  varied in different numerical experiments. We used no-flux boundary conditions for the spatial variable. Inasmuch as we wanted to study the behavior of the traveling wave solutions in an infinite space we chose such space interval so that to avoid the influence of boundaries.

We used an explicit difference scheme. The approximation of the taxis term is an "up-wind" explicit scheme [25] which is frequently used for cross-diffusion systems (e.g., [26]). More precisely,

$$\begin{aligned} u_i^{t+1} &= u_i^t + \frac{\Delta t}{(\Delta x)^2} (u_{i+1}^t - 2u_i^t + u_{i-1}^t) - \frac{\Delta t}{(\Delta x)^2} (A(v_{i+1}^t - v_i^t) - B(v_i^t - v_{i-1}^t)), \\ v_i^{t+1} &= v_i^t + (\Delta t)g(u_i^t, v_i^t), \quad i = 2, \dots, N-1, \end{aligned}$$

where for the positive taxis (pursuit) (i.e.,  $f(u, v) < 0$ ),

$$\begin{aligned} A &= f(u_i^t, v_i^t) \quad \text{if } v_{i+1} \geq v_i, \\ A &= f(u_{i+1}^t, v_{i+1}^t) \quad \text{if } v_{i+1} < v_i, \\ B &= f(u_{i-1}^t, v_{i-1}^t) \quad \text{if } v_i \geq v_{i-1}, \\ B &= f(u_i^t, v_i^t) \quad \text{if } v_i < v_{i-1}. \end{aligned}$$

For the negative taxis (invasion):

$$\begin{aligned} A &= f(u_i^t, v_i^t) \quad \text{if } v_{i+1} < v_i, \\ A &= f(u_{i+1}^t, v_{i+1}^t) \quad \text{if } v_{i+1} \geq v_i, \\ B &= f(u_{i-1}^t, v_{i-1}^t) \quad \text{if } v_i < v_{i-1}, \\ B &= f(u_i^t, v_i^t) \quad \text{if } v_i \geq v_{i-1}. \end{aligned}$$

We used  $\Delta t = 0.001$ ,  $\Delta x = 0.1$ . For the boundary conditions:

$$\begin{aligned} u_1^{t+1} &= u_2^t, & u_N^{t+1} &= u_{N-1}^t, \\ v_1^{t+1} &= v_2^t, & v_N^{t+1} &= v_{N-1}^t. \end{aligned}$$

For the initial conditions we used numerical solutions of the corresponding wave systems.

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